Richardson-Lucy Algorithm for image restoration

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This document explains Richardson-Lucy algorithm Lucy [1974]; Richardson [1972] which is used for image restoration.

1 Imaging equation

Assuming Poisson noise, imaging equation is derived as

$$g(x) = Poisson(f(x) \otimes h(x)), \tag{1}$$

where g, f, and h are a blurred image, the latent image, and the Point Spread Function (PSF) respectively. $Poisson$ operator adds Poisson noise to the image.

2 Bayes' theorem

Given a blurred image g, a posterior distribution is

$$P(f(x)|g(x)) = \frac{P(g(x)|f(x))P(f(x))}{P(g(x))}, \tag{2}$$

where $P(g(x)|f(x))$, $P(g(x))$, and $P(f(x))$ are the likelihood, the evidence, and the prior distribution.

3 Poisson noise likelihood

When image noise follows Poisson distribution, the likelihood is formulated as

$$P\left(g(x)|f(x)\right) = \prod_x \frac{f(x) \otimes h(x)^{g(x)} \exp\left(-f(x) \otimes h(x)\right)}{g(x)!}. \tag{3}$$

Maximization of the likelihood is equivalent to minimization of its negative logarithm. Therefore, the negative log likelihood $L(f(x))$ is

$$\begin{aligned} L\left(f(x)\right) &= -\ln\left(P\left(g(x)|f(x)\right)\right) && (4)\\ &= \sum_x \ln \frac{f(x)\otimes h(x)^{g(x)}\exp\left(-f(x)\otimes h(x)\right)}{g(x)!} && (5)\\ &= \sum_x f(x)\otimes h(x) - g(x)\ln\left(f(x)\otimes h(x)\right) + \ln g(x)!. && (6) \end{aligned}$$

Consider a small perturbation Δx. The negative log likelihood $L(f(x+\Delta x))$ is

$$L(f(x+\Delta x)) = \sum_x f(x+\Delta x)\otimes h(x) - g(x)\ln\left(f(x+\Delta x)\otimes h(x)\right) + \ln g(x)!. \quad (7)$$

Removing constant term w.r.t. f, we obtain

$$L(f(x+\Delta x)) = \sum_x f(x+\Delta x)\otimes h(x) - g(x)\ln\left(f(x+\Delta x)\otimes h(x)\right). \quad (8)$$

Assuming $f(x+\Delta x) = f(x) + f(\Delta x)$,

$$\begin{aligned} L(f(x+\Delta x)) &= \sum_x (f(x)+f(\Delta x))\otimes h(x) - g(x)\ln\left((f(x)+f(\Delta x))\otimes h(x)\right)\\ &= \sum_x f\otimes h(x) + f\otimes h(\Delta x) - g(x)\ln\left(f\otimes h(x) + f\otimes h(\Delta x)\right). && (9) \end{aligned}$$

For simplicity,

$$\begin{aligned} f\otimes h(x) &= f(x)\otimes h(x),\\ f\otimes h(\Delta x) &= f(\Delta x)\otimes h(x). \end{aligned}$$

$$\begin{aligned} L(f(x+\Delta x)) &= \sum_x f\otimes h(x) + f\otimes h(\Delta x) - g(x)\ln\left(f\otimes h(x) + f\otimes h(\Delta x)\right)\\ &= \sum_x f\otimes h(x) + f\otimes h(\Delta x) - g(x)\ln\left[f\otimes h(x)\left(1+\frac{f\otimes h(\Delta x)}{f\otimes h(x)}\right)\right]\\ &= \sum_x f\otimes h(x) + f\otimes h(\Delta x) - g(x)\ln\left(f\otimes h(x)\right) - g(x)\ln\left(1+\frac{f\otimes h(\Delta x)}{f\otimes h(x)}\right)\\ &= L(f(x)) + \sum_x f\otimes h(\Delta x) - g(x)\ln\left(1+\frac{f\otimes h(\Delta x)}{f\otimes h(x)}\right). \end{aligned}$$

Following Taylor expansion $\ln(1+x) \approx x - \frac{x^2}{2}$,

$$\begin{aligned} L(f(x+\Delta x)) &= L(f(x)) + \sum_x f\otimes h(\Delta x) - g(x)\ln\left(1+\frac{f\otimes h(\Delta x)}{f\otimes h(x)}\right)\\ &= L(f(x)) + \sum_x f\otimes h(\Delta x) - g(x)\frac{f\otimes h(\Delta x)}{f\otimes h(x)} + \frac{1}{2}g(x)(\frac{f\otimes h(\Delta x)}{f\otimes h(x)})^2. \end{aligned}$$

Omitting the last term, because it's too small,

$$
\begin{aligned}
L(f(x+\Delta x)) &= L(f(x)) + \sum_x f \otimes h(\Delta x) - g(x)\frac{f \otimes h(\Delta x)}{f \otimes h(x)} \\
&= L(f(x)) + \sum_x f \otimes h(\Delta x)\left(1 - \frac{g(x)}{f \otimes h(x)}\right).
\end{aligned}
$$

From the definition of convolution integral $\int ab \otimes cdx = \int ba \otimes \overline{h}dx$, where $\overline{h}h$ is the adjoint of h,

$$
\begin{aligned}
L(f(x+\Delta x)) &= L(f(x)) + \sum_x f \otimes h(\Delta x)\left(1 - \frac{g(x)}{f \otimes h(x)}\right) \\
&= L(f(x)) + \sum_x f\left(1 - \frac{g(x)}{f \otimes h(x)}\right) \otimes \overline{h}(\Delta x).
\end{aligned}
$$

The partial derivative of $L(f(x))$ on x is derived as

$$
\begin{aligned}
\frac{\partial L(f(x))}{\partial x} &= \frac{L(f(x+\Delta x)) - L(f(x))}{\Delta x} \\
&= \frac{1}{\Delta x}\sum_x f\left(1 - \frac{g(x)}{f \otimes h(x)}\right) \otimes \overline{h}.
\end{aligned}
$$

Since the minimization of the negative log likelihood is obtained by finding x satisfying

$$
\frac{\partial L(f(x))}{\partial x} = 0.
$$

Thus, we have

$$
\begin{aligned}
\left(1 - \frac{g(x)}{f \otimes h(x)}\right) \otimes \overline{h} = 0 \\
1 - \frac{g(x)}{f \otimes h(x)} \otimes \overline{h} = 0
\end{aligned}
\tag{10}
$$

Using the convergence condition $\frac{f^{n+1}}{f^n} = 1$, we obtain the update rule as

$$
\frac{f^{n+1}}{f^n} = \frac{g(x)}{f \otimes h(x)} \otimes \overline{h}. \tag{11}
$$

Finally, we obtain the Richardson-Lucy deconvolution algorithm as

$$
f^{n+1} = \left(\frac{g(x)}{f \otimes h(x)} \otimes \overline{h}\right) f^n. \tag{12}
$$

Bibliography

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