Richardson-Lucy Algorithm with Gaussian noise

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This document extends Richardson-Lucy algorithm Lucy [1974]; Richardson [1972] to handle Gaussian noise [1].

1 Imaging equation

Assuming Gaussian noise, imaging equation is derived as

$$g(x) = f(x) \otimes h(x) + n(x), \tag{1}$$

where g, f, h, and$n$ are a blurred image, the latent image, the Point Spread Function (PSF), and image noise respectively.

2 Bayes' theorem

Given a blurred image g, a posterior distribution is

$$P(f(x)|g(x)) = \frac{P(g(x)|f(x))P(f(x))}{P(g(x))}, \tag{2}$$

where $P(g(x)|f(x))$, $P(g(x))$, and $P(f(x))$ are the likelihood, the evidence, and the prior distribution.

3 Gaussian noise likelihood

When image noise follows Gaussian distribution, the likelihood is formulated as

$$\begin{aligned} P\left(g(x)|f(x)\right) &= \prod_x N(f \otimes h(x), \sigma^2) \\ &= \prod_x \exp\left(\frac{-|g(x) - f \otimes h(x)|^2}{2\sigma^2}\right) \end{aligned} \tag{3}$$

Maximization of the likelihood is equivalent to minimization of its negative logarithm. Therefore, the negative

[1] Original Richardson-Lucy algorithm assumes Poisson noise

log likelihood $L(f(x))$ is

$$
\begin{aligned}
L\left(f(x)\right) &= -\ln\left(P\left(g(x)|f(x)\right)\right) && (4)\\
&= \sum_x \left(g(x) - f \otimes h(x)\right)^2. && (5)
\end{aligned}
$$

Consider a small perturbation Δx. The negative log likelihood $L(f(x+\Delta x))$ is

$$
\begin{aligned}
L(f(x+\Delta x)) &= \sum_x g(x)^2 - 2g(x) f \otimes h(x+\Delta x) + f \otimes h(x+\Delta x)^2 \\
&= L(f(x)) + \sum_x -2g(x) f \otimes h(\Delta x) + 2 f \otimes h(x) f \otimes h(\Delta x) + f \otimes h(\Delta x)^2
\end{aligned}
$$

Omitting the last term, because it's too small,

$$
\begin{aligned}
L(f(x+\Delta x)) &= L(f(x)) + \sum_x -2g(x) f \otimes h(\Delta x) + 2 f \otimes h(x) f \otimes h(\Delta x) + f \otimes h(\Delta x)^2 \\
&= L(f(x)) + \sum_x -2g(x) f \otimes h(\Delta x) + 2 f \otimes h(x) f \otimes h(\Delta x) \\
&= L(f(x)) + 2\sum_x f \otimes h(\Delta x) \left(f \otimes h(x) - g(x)\right)
\end{aligned}
$$

From the definition of convolution integral $\int ab \otimes c dx = \int ba \otimes \overline{h} dx$, where $\overline{h}h$ is the adjoint of h,

$$
\begin{aligned}
L(f(x+\Delta x)) &= L(f(x)) + 2\sum_x f \otimes h(\Delta x) \left(f \otimes h(x) - g(x)\right) \\
&= L(f(x)) + 2\sum_x f \left(f \otimes h(x) - g(x)\right) \otimes \overline{h}(\Delta x)
\end{aligned}
\tag{6}
$$

The partial derivative of $L(f(x))$ on x is derived as

$$
\begin{aligned}
\frac{\partial L(f(x))}{\partial x} &= \frac{L(f(x+\Delta x)) - L(f(x))}{\Delta x} \\
&= \frac{2}{\Delta x} \sum_x f \left(f \otimes h(x) - g(x)\right) \otimes \overline{h}.
\end{aligned}
$$

Since the minimization of the negative log likelihood is obtained by finding x satisfying

$$
\frac{\partial L(f(x))}{\partial x} = 0.
$$

Thus, we have

$$
f\left(f \otimes h(x) - g(x)\right) \otimes \overline{h} = 0. \tag{7}
$$

Using the convergence condition $f^{n+1} - f^n \approx 0$, we obtain the update rule as

$$
f^{n+1} - f^n = \left(f \otimes h(x) - g(x)\right) \otimes \overline{h}. \tag{8}
$$

Finally, we obtain the Richardson-Lucy deconvolution algorithm as

$$
f^{n+1} = f^n + \left(g(x) - f^n \otimes h(x)\right) \otimes \overline{h}. \tag{9}
$$

Bibliography

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