Eigen vector

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Prerequisities

Let $\mathbf{b} = \mathbf{A}\mathbf{x}$ be a linear transformation system of n dimension.
A square matrix $\mathbf{A}$ is a function that transforms an n dimensional vector $\mathbf{x}$ to another n dimensional vector $\mathbf{b}$.

Eigenvector

Eigen vector $\mathbf{x}$ is a non-zero vector that is imaged into a vector $\lambda\mathbf{x}$ by $\mathbf{A}$ as

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \tag{1}$$

where the scalar λ is called eigen value.

Eigenvector (cont.)

Eq. (1) is equivalent to

$$\lambda \mathbf{x} - \mathbf{A}\mathbf{x} = \mathbf{0} \text{ or } \lambda \mathbf{I}\mathbf{x} - \mathbf{A}\mathbf{x} = \mathbf{0}$$

that is

$$(\lambda \mathbf{I} - \mathbf{A})\,\mathbf{x} = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{m1} & -a_{m2} & \cdots & \lambda - a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \mathbf{0}. \qquad (2)$$

This equation is homogeneous!

Homogeneous system

Homogeneous systems form as

$$\mathbf{Ax} = \mathbf{0}.$$

Homogeneous systems have at least trivial solution

$$\mathbf{x} = \mathbf{0}$$

A non-trivial solution is any solution that

$$\mathbf{x} \neq \mathbf{0}.$$

Homogeneous systems have non-trivial solutions if and only if the determinant of the coefficient matrix is equal to zero as

$$\det(\mathbf{A}) = |\mathbf{A}| = 0.$$

Characteristic equation

Non-trivial solutoin of Eq. (2) should satisfy

$$\det(\lambda \mathbf{I} - \mathbf{A}) = |\lambda \mathbf{I} - \mathbf{A}| = 0. \tag{3}$$

The expansion of the equation forms as

$$\phi(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0. \tag{4}$$

This equation is called characteristic equation of matrix **A**.

Eigenvalues and eigenvectors computation

1. Compute eigenvalues $\{\lambda_i\}$ by solving Eq. (3).
2. Compute eigenvector $\mathbf{x}_i$ by solving Eq. (2) for each eigenvalue λ_i.